Analytical calculations of numerical dissipation

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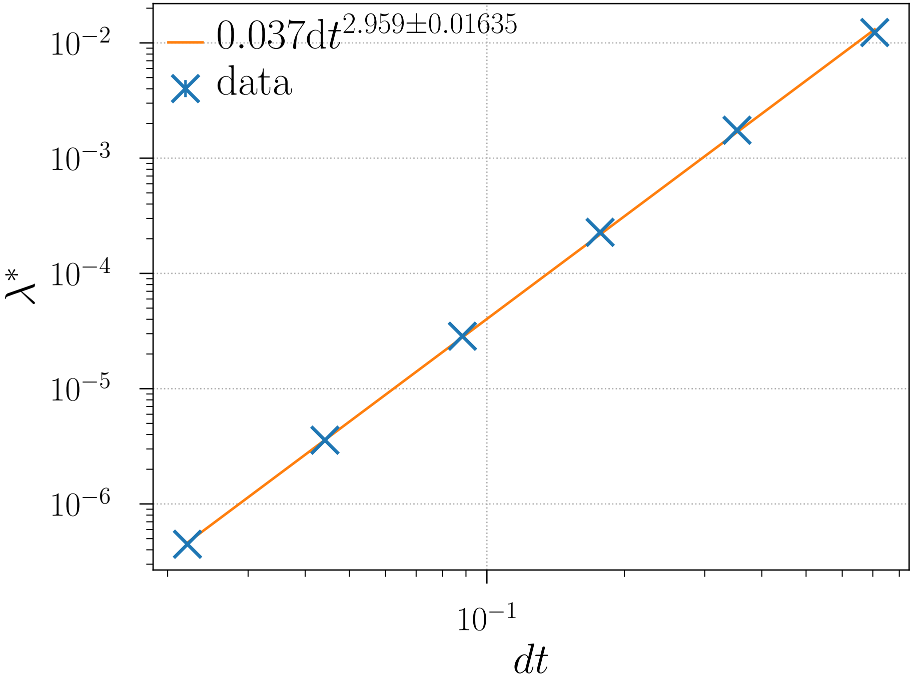

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What is Numerical Dissipation?

Everyone is talking about it,
but no one knows what is really is.

Numerical Experiments

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On the Measurements of Numerical Viscosity and Resistivity in Eulerian MHD Codes

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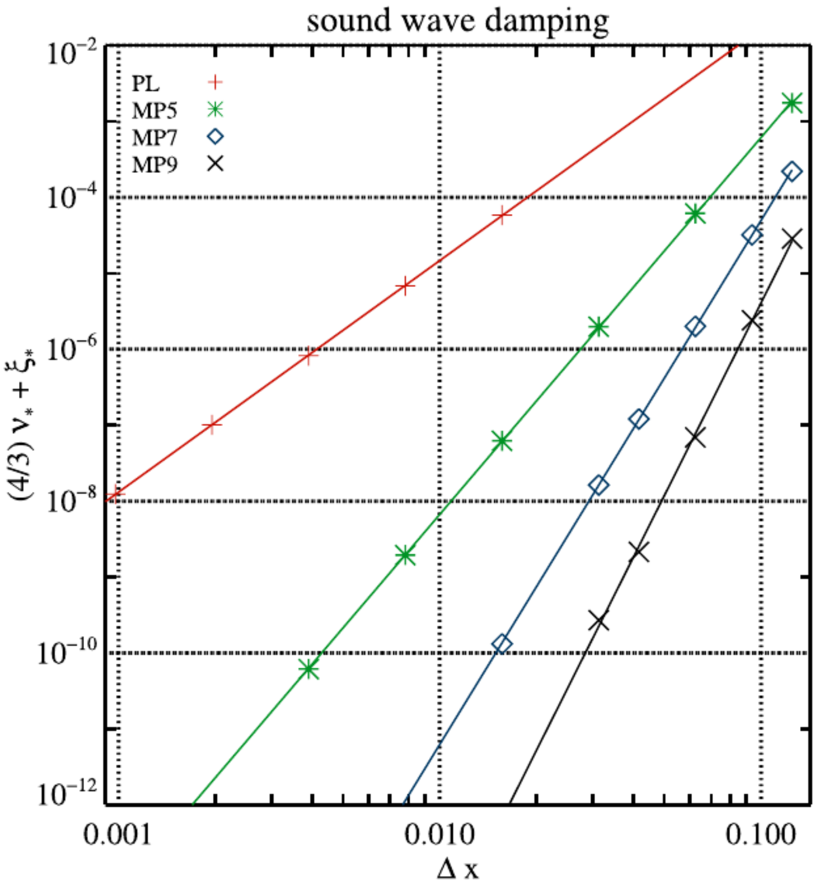
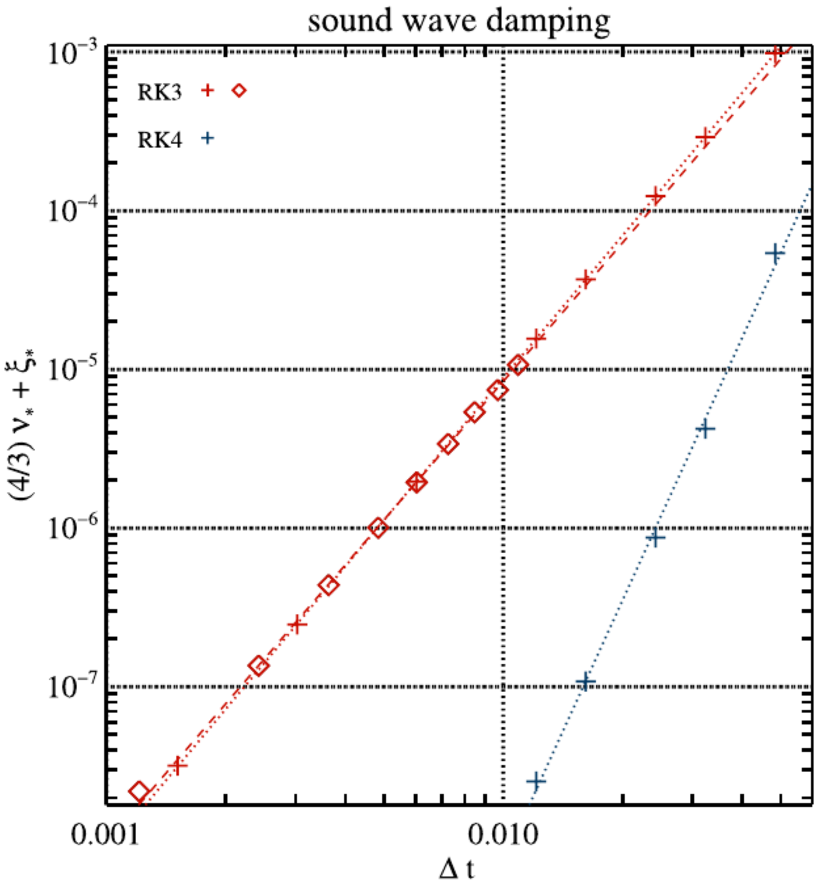
Table 1
Wave Damping Simulations I

Series	Wave	Reco	Riemann	Time	CFL	Resolution	$\mathfrak{N}^{\Delta x}_{\mathrm{tot}}$	r	$\mathfrak{N}^{\Delta t}_{\mathrm{tot}}$	q
#S1	sound	PL	HLL	RK4	0.01	64...1028	14.3 ± 0.7	3.049 ± 0.009	···	···
#S2	sound	MP5	LF	RK4	0.01	8...256	42.9 ± 2.3	4.957 ± 0.013	···	···
#S3	sound	MP5	HLL	RK4	0.01	8...256	43.4 ± 2.5	4.961 ± 0.014	···	···
#S4	sound	MP5	HLLD	RK4	0.01	8...256	42.7 ± 2.2	4.956 ± 0.013	···	···
#S5	sound	MP7	HLL	RK4	0.01	8...64	302 ± 20	6.897 ± 0.021	···	···
#S6	sound	MP9	HLL	RK4	0.01	8...32	830 ± 340	8.42 ± 0.15	···	···
#S7	sound	MP9	HLL	RK3	0.5	8...256	···	···	1.492 ± 0.013	2.985 ± 0.002
#S8	sound	MP9	HLL	RK3	0.1...0.9	64	···	···	2.45 ± 0.17	2.95 ± 0.01
#S9	sound	MP9	HLL	RK4	0.5	8...32	···	···	71 ± 32	5.5 ± 0.2
#A1	Alfvén	MP5	LF	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	···	···
#A2	Alfvén	MP5	HLL	RK4	0.01	8...256	42.6 ± 2.1	4.96 ± 0.01	···	···
#A3	Alfvén	MP5	HLLD	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	···	···
#A4	Alfvén	MP7	HLL	RK4	0.01	8 ...128	44 ± 53	6.19 ± 0.03	···	···
#A5	Alfvén	MP9	HLL	RK4	0.01	8...64	1190 ± 190	8.57 ± 0.06	···	···
#A6	Alfvén	MP9	HLL	RK3	0.8	16...128	···	···	0.86 ± 0.08	2.949 ± 0.022
#A7	Alfvén	MP9	HLL	RK4	0.8	8...64	···	···	7.6 ± 2.5	5.18 ± 0.10
#A8	Alfvén	MP5	HLL	RK3	0.5	5...1024	···	···	···	···
#MS1	magnetosonic	MP5	HLL	RK4	0.01	8...128	40 ± 3	4.95 ± 0.02	···	···
#MS2	magnetosonic	MP7	HLL	RK4	0.01	8...64	288 ± 20	6.903 ± 0.023	···	···
#MS3	magnetosonic	MP9	HLL	RK4	0.01	8...32	1970 ± 160	8.82 ± 0.03	···	···
#MS4	magnetosonic	MP9	HLL	RK3	0.1...0.9	64	···	···	1.77 ± 0.06	2.977 ± 0.007
#MS5	magnetosonic	MP9	HLL	RK4	0.2...0.9	64	···	···	4.3 ± 0.8	4.834 ± 0.013

Numerical Experiments





Local Truncation Error

discretized

exact $\mathcal{L}[\hat{u}] = 0$

Definition 7.4. *The quantity*

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = \mathcal{A}_{(h)}[\hat{u}] = A_{(h)}\hat{u} - F_{(h)}.$$

is called the local truncation error (local residual) of the numerical scheme $\mathcal{A}_{(h)}[\,] = 0$.

Example 7.5. *Find the local truncation error of the numerical scheme*

$$\mathcal{A}_{(h)}[u] = \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} - f_k = 0$$

for the solution of

$$u'' - f = 0.$$

This is solved exactly.

Solution. Now

$$\mathcal{A}_{(h)}[\hat{u}] = \frac{\hat{u}_{k-1} - 2\hat{u}_k + \hat{u}_{k+1}}{h^2} - f_k = (\hat{u}_k'' + O(h^2)) - f_k,$$

but

$$\mathcal{L}[\hat{u}] = \hat{u}_k'' - f_k = 0,$$

so using the definition directly

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = O(h^2).$$

Navier-Stokes

$$\frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\tau}_\rho$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -c_\mathrm{S}^2\boldsymbol{\nabla}\ln\rho + \boldsymbol{F}_\mathrm{visc} + \boldsymbol{\tau}_{\boldsymbol{u}}$$

numerical viscosity: $\boldsymbol{\tau}_{\boldsymbol{u}} = \nu^*\boldsymbol{\nabla}^2\boldsymbol{u} + \ldots$

$$E_\mathrm{kin} = \frac{1}{2}\int_V \rho\boldsymbol{u}^2 \, \mathrm{d}V$$

$$\partial_t E_\mathrm{kin} = \frac{1}{2}\int_V -\rho\boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) - c_\mathrm{s}^2\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\ln(\rho) + \rho\boldsymbol{u}\cdot\boldsymbol{F}_\mathrm{visc} - \rho\boldsymbol{u}^2\boldsymbol{u}\cdot\boldsymbol{\nabla}\ln(\rho) - \rho\boldsymbol{u}^2\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}V$$
$$- \int_V \rho\boldsymbol{u}\cdot\boldsymbol{\tau}_{\boldsymbol{u}} + \frac{1}{2}\rho\boldsymbol{\tau}_\mathrm{lnrho}\boldsymbol{u}^2 \, \mathrm{d}V$$

How does the numerical error depend on the grid resolution and time integration?

Analytical Approach

Discretize space onto a grid.

Use finite difference stencils for derivatives:

$$\partial_x f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

Method of lines for every (i,j): $\partial_t \boldsymbol{f} = \boldsymbol{g}(\boldsymbol{f})$

Time integration (Runge-Kutta):

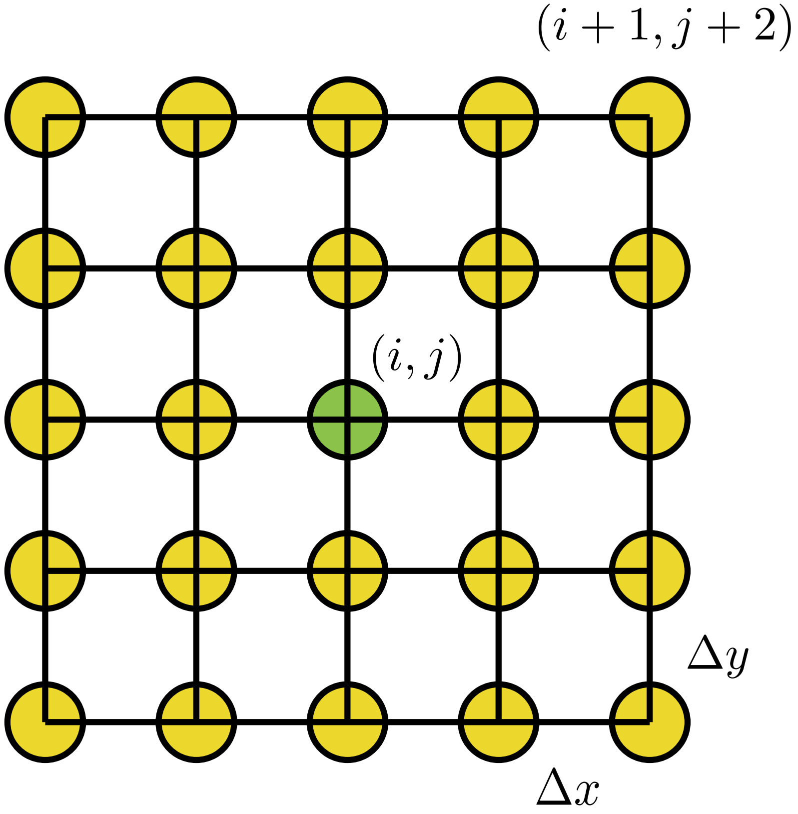
$$\boldsymbol{k}_1 = \Delta t \boldsymbol{g}(\boldsymbol{f}(t)) \quad \boldsymbol{k}_2 = \Delta t \boldsymbol{g}(\boldsymbol{f}(t) + \boldsymbol{k}_1/2)$$

$$\boldsymbol{f}^{n+1} = \boldsymbol{f}^n + \boldsymbol{k}_2$$

Time Taylor expansion:

$$\tilde{\boldsymbol{f}}^{n+1} = \boldsymbol{f}^n + \Delta t \boldsymbol{g}(\boldsymbol{f}^n) + \Delta t^2/2 \boldsymbol{g}(\boldsymbol{f}^n)\boldsymbol{g}'(\boldsymbol{f}^n) + \dots$$

Space Taylor expansion for off-diagonals.



$$\boldsymbol{\tau} = \tilde{\boldsymbol{f}}^{n+1} - \boldsymbol{f}^{n+1}$$

Analytical Approach

Linear acoustic wave:

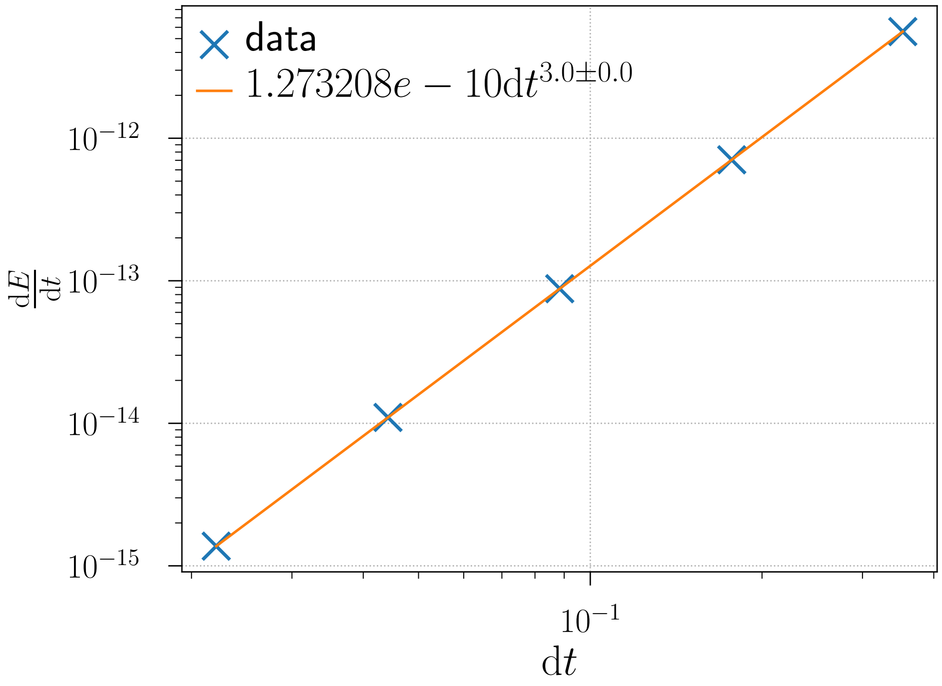
$$\rho(x) = \rho_0 + \delta \sin(x)$$

$$u(x) = c_s \delta \sin(x)$$

$$x \in [0, 2\pi]$$

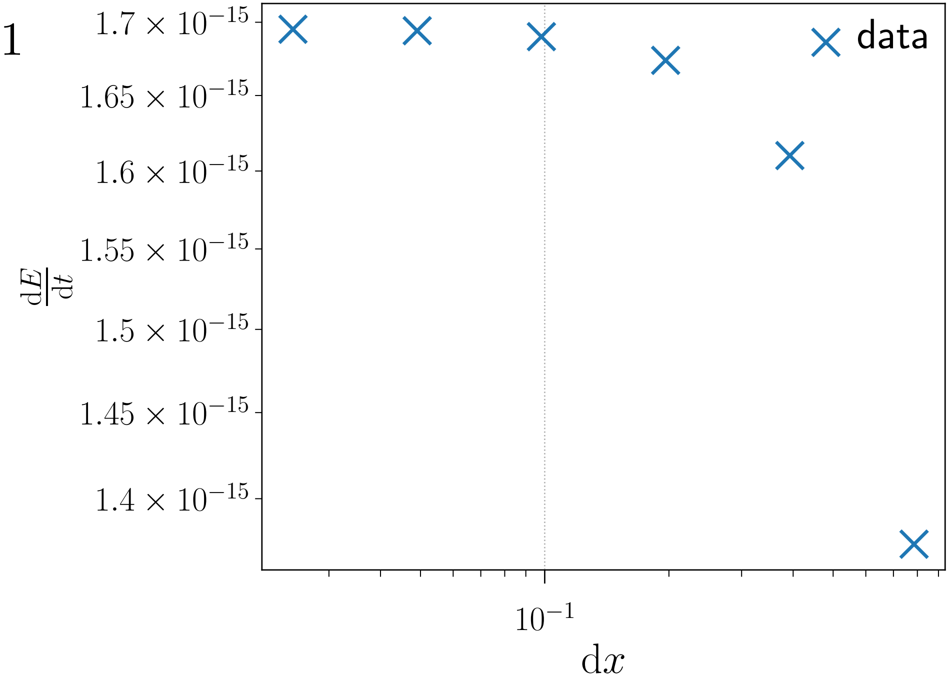
$$n_x = 8(\mathrm{d}x = 0.78539816)$$

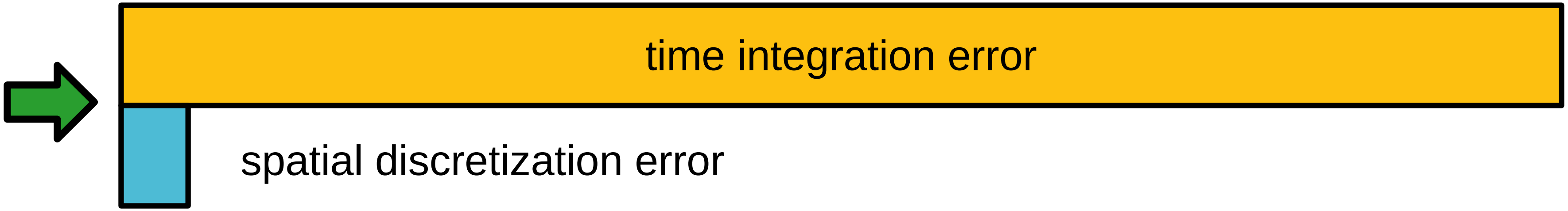
4th order Runge-Kutta



Analytical Approach

$\mathrm{d}t = 0.0221$





Acoustic Wave Damping

wave propagation: $e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega t}$

$\mathrm{Im}(\omega) \neq 0$ ⇨ wave damping

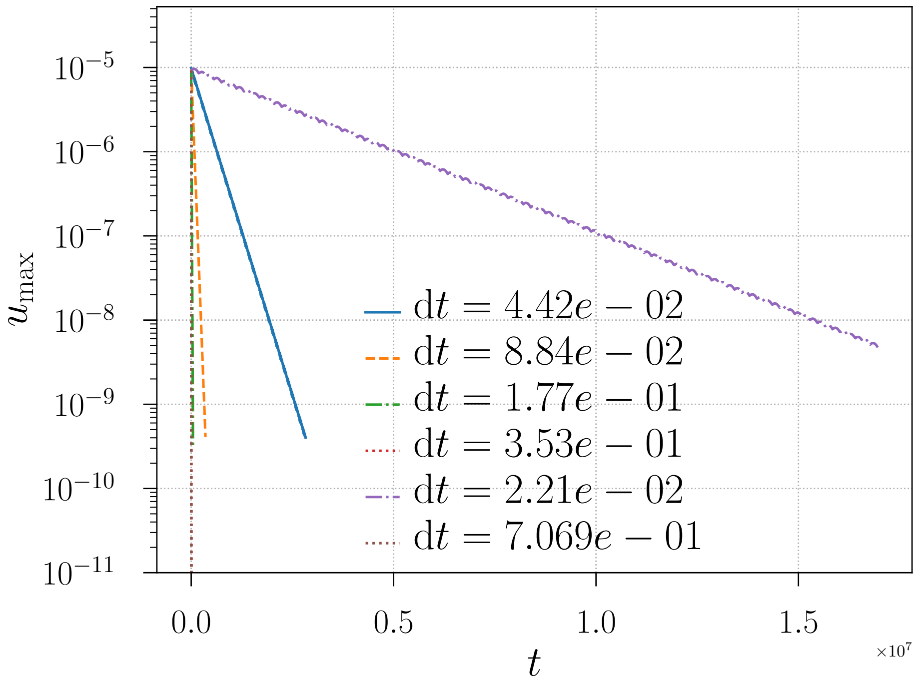
damping parameter: $\lambda = \mathrm{Im}(\omega) = \frac{1}{2}\nu k^2$

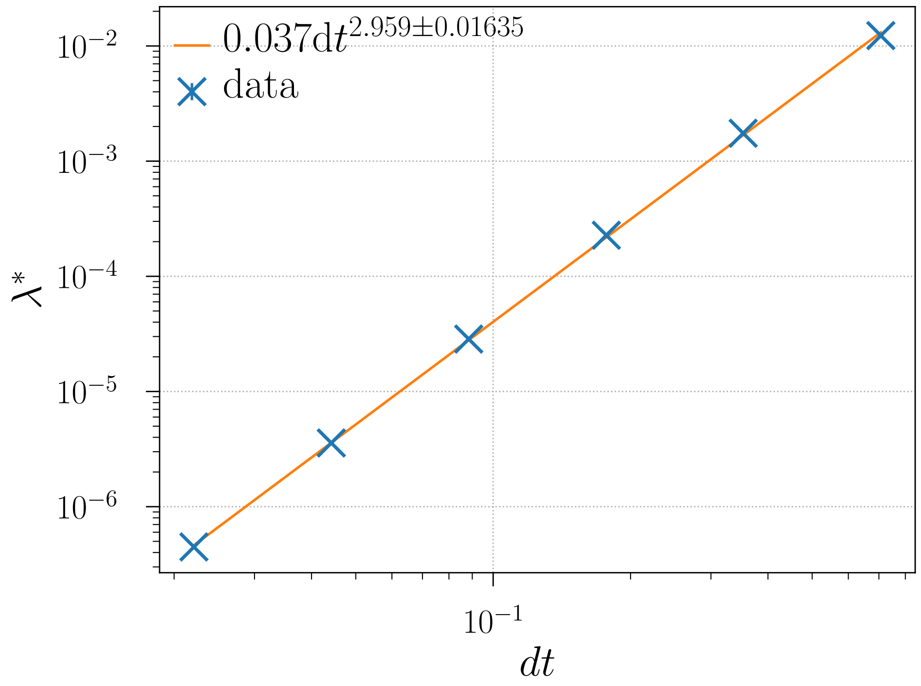
numerical damping: $\lambda^* = \frac{1}{2}\nu^* k^2$

⇨ Perform linear acoustic wave simulations and measure the numerical damping.

$n \in [8, 256]$

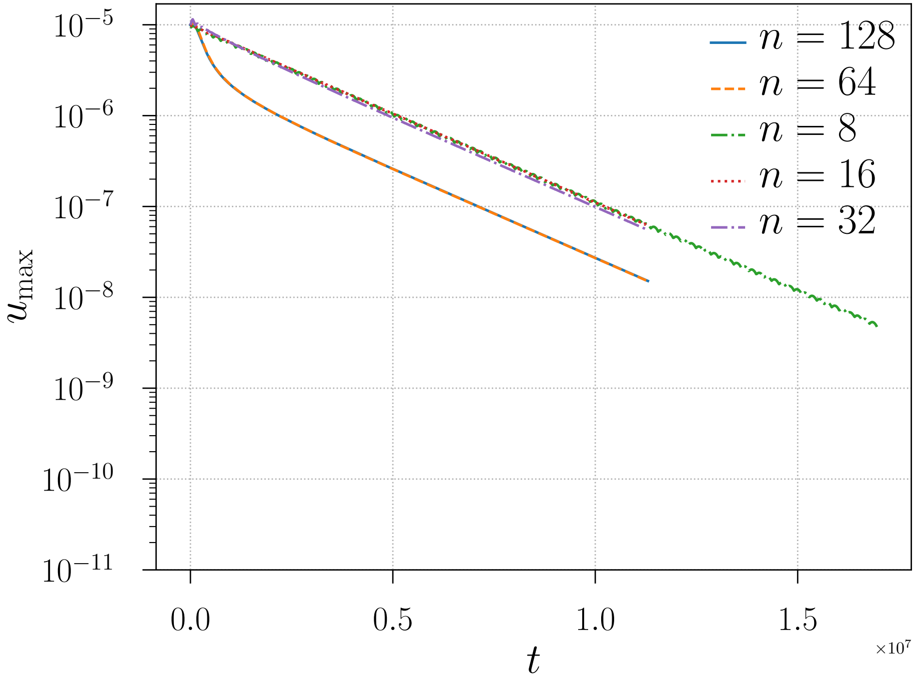
Finite Difference: Pencil Code

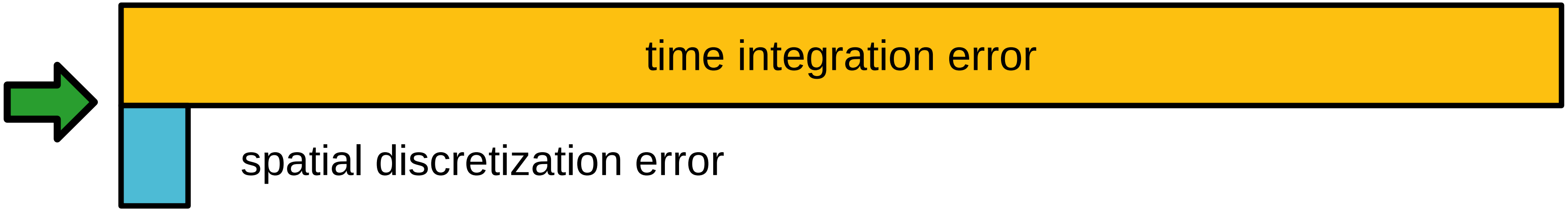




Finite Difference: Pencil Code

Fix time step.





Alfvén Wave Damping

wave propagation: $e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega t}$

$\mathrm{Im}(\omega) \neq 0$ ⇨ wave damping

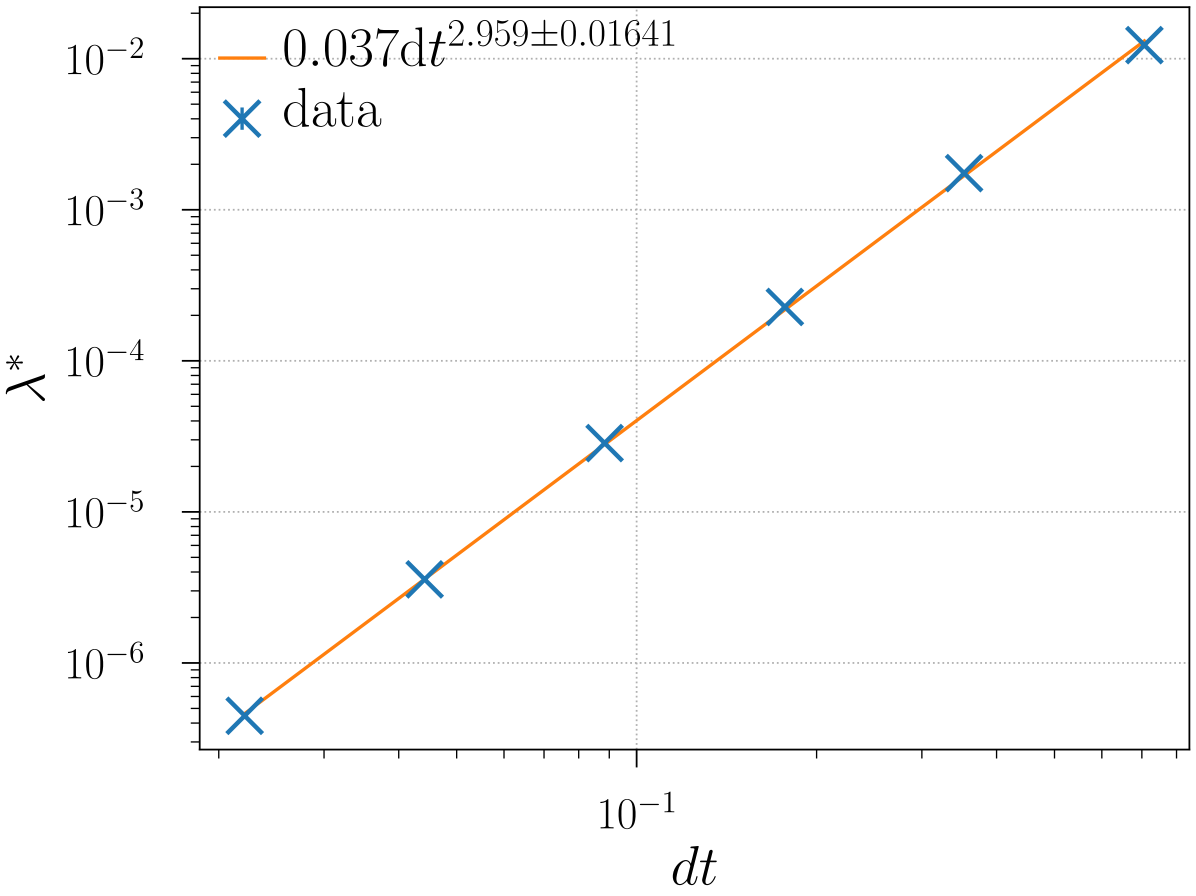
damping parameter: $\lambda = \mathrm{Im}(\omega) = \frac{1}{2}(\nu + \eta)k^2$

numerical damping: $\lambda^* = \frac{1}{2}(\nu^* + \eta^*)k^2$

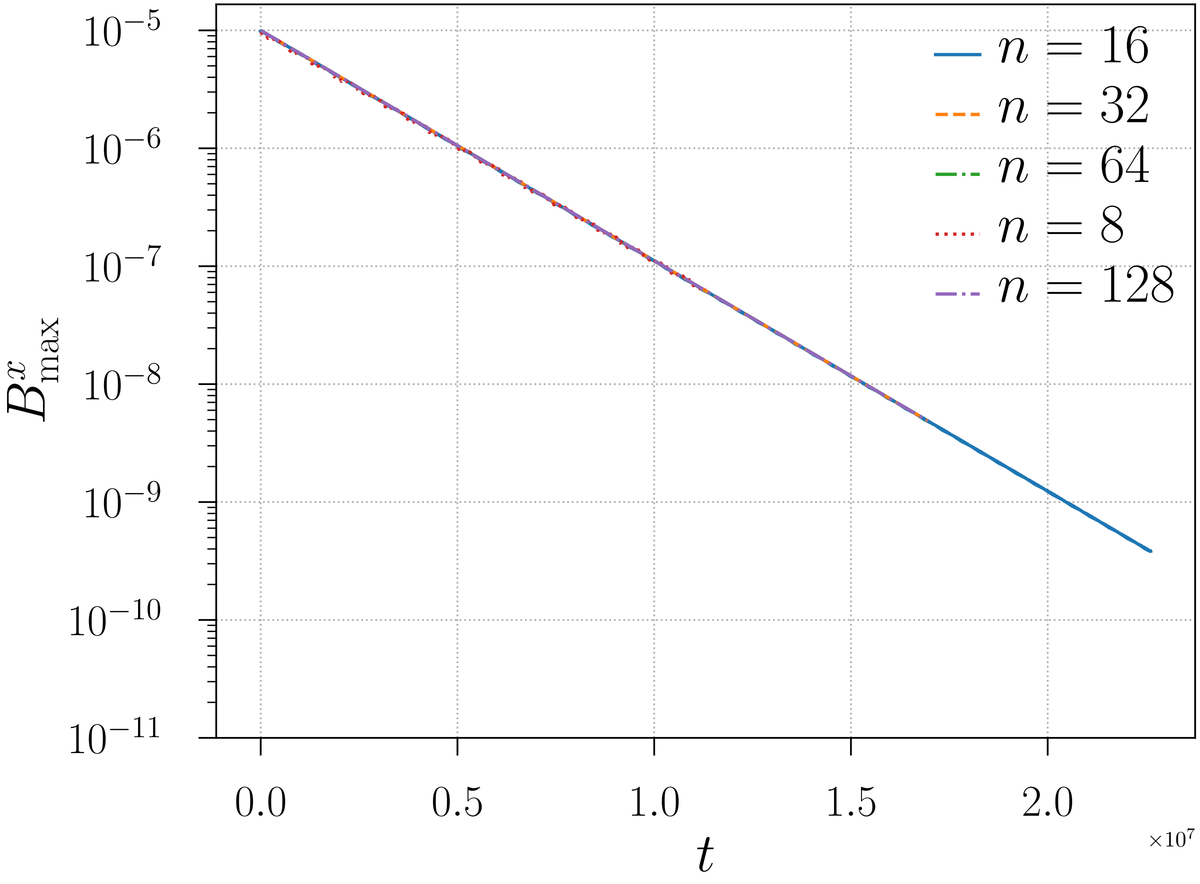
⇨ Perform linear Alfven wave simulations and measure the numerical damping.

$n \in [8, 256]$

Alfvén Wave Damping

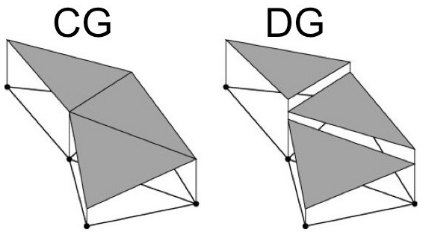


Alfvén Wave Damping



Trixi.jl

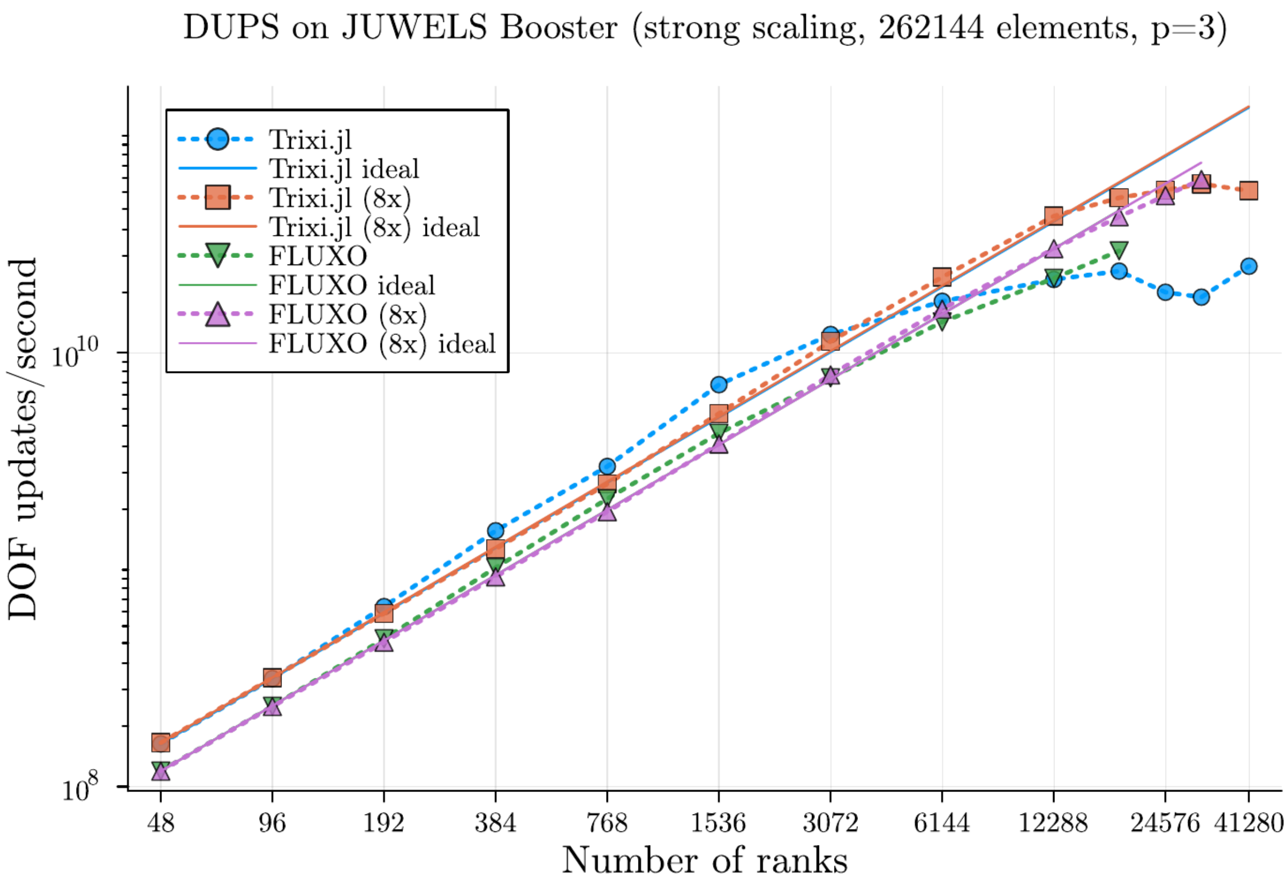
Discontinuous Galerkin



(Stack Overflow)

julia

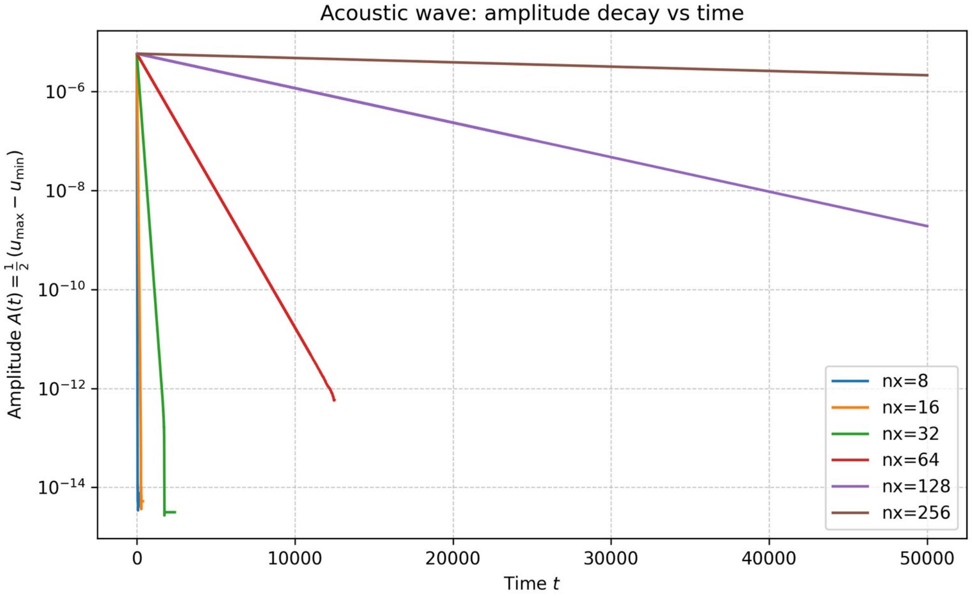
OrdinaryDiffEq (SciML)

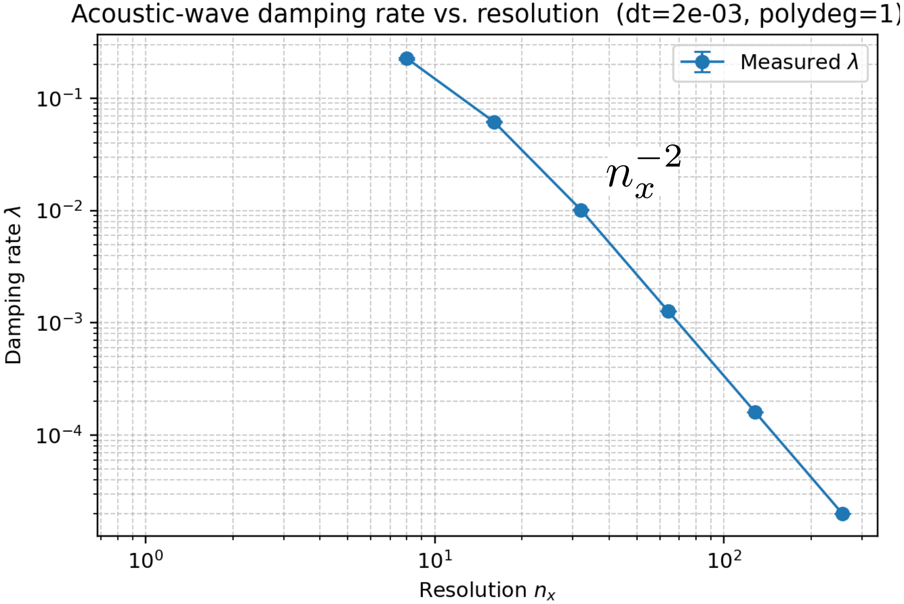


Trixi.jl

$$\mathrm{d}t = 2 \times 10^{-3} \qquad \text{polydeg} = 1$$

Integrator: `CarpenterKennedy2N54`



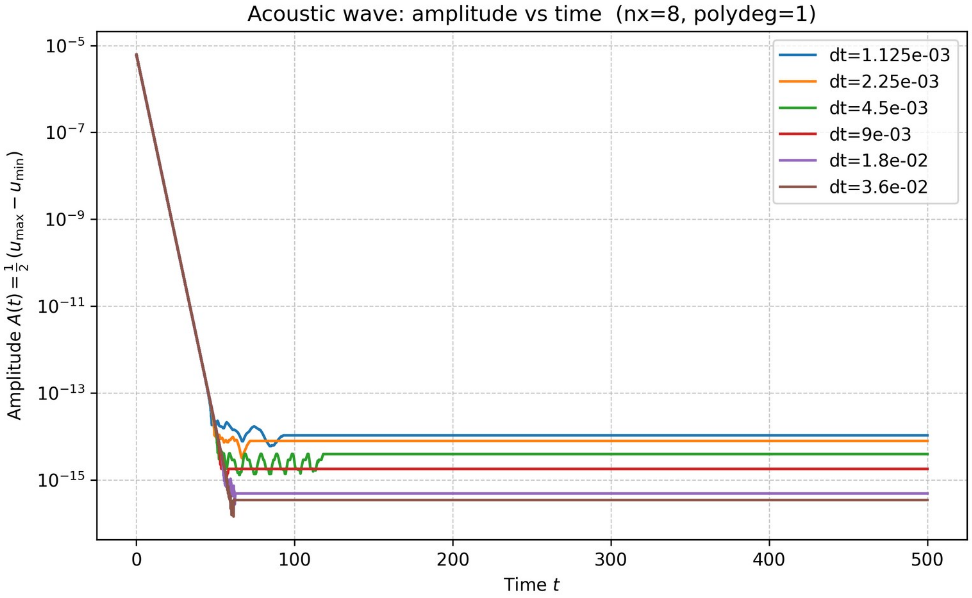


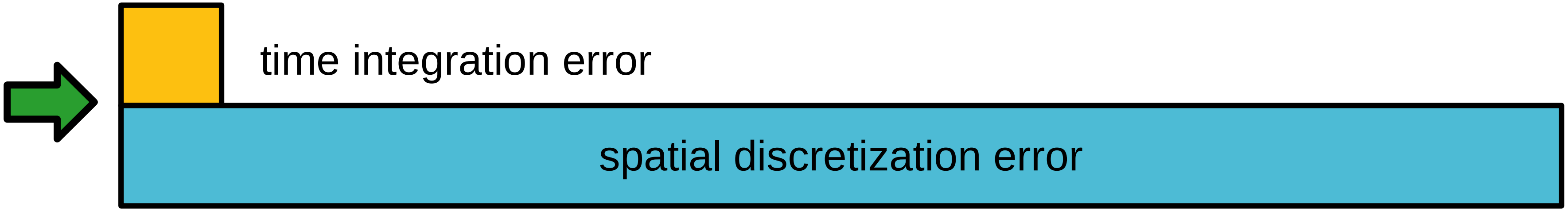
Trixi.jl

$n_x = 8 \qquad \text{polydeg} = 1$

Integrator: `CarpenterKennedy2N54`

No apparent dependence on $\mathrm{d}t$.



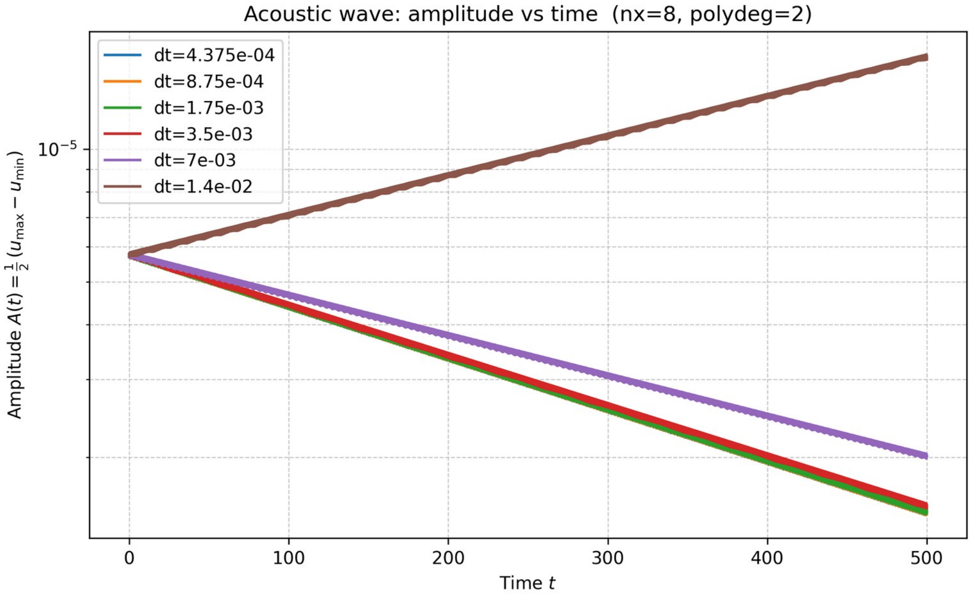


Trixi.jl

$n_x = 8 \qquad \text{polydeg} = 2$

2nd order integrator: `SSPRK22`

Negative effective damping.



time integration error

spatial discretization error

Conclusion

- Numerical viscosity and diffusion can be calculated analytically.
- Find numerical diffusion in numerical experiments.
- Dependence on time step and resolution.

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